CONFORMAL TRANSFORMATIONS OF PERIOD n AND GROUPS GENERATED BY THEM.*

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Introduction.

Professor Kasner† has discussed the characteristics of groups of transformations generated by conformal transformations of period 2. He considered both the "direct" and what he terms "reverse," or "improper" transformations. The former type may be represented in the form $Z = f(z)$, where $f(z)$ is analytic at the origin, and converts it into itself. The latter type, termed a "symmetry," may be represented in the form $Z = f(z_0)$, where z_0 is the conjugate of z.

In discussing direct transformations, Professor Kasner utilizes the implicit form

$$Z + z = d_2(Z - z)^2 + d_4(Z - z)^4 + \cdots,$$

which includes every transformation of period 2, and every solution of which, for arbitrary values of the coefficients d, yields a transformation of period 2. In considering the conditions under which a given transformation

$$Z = c_1z + c_2z^2 + c_3z^3 + \cdots$$

can be factored as the product of two transformations of period 2, Kasner obtains the condition $c_3 - c_2^2 = 0$, besides the obvious condition $c_1 = 1$. If $c_2 = 0$, further conditions must be satisfied.

In the case of reverse transformations, the given transformation can always be factored into two "symmetries" if $|c_1| = 1$, and the angle of c_1 is incommensurable with π. If the latter condition is not satisfied, then the given transformation can in any case be factored into four symmetries.

It is the purpose of this paper to generalize the results obtained by Kasner to include transformations of period n. In the case of direct transformations, it is remarkable that no such necessary condition as that obtained by Kasner upon the coefficients following the first is found necessary for factorization into transformations of periods greater than 2. In the case $c_1 = 1$ there is, however, a non-zero relation between the coefficients immediately following the first in the given transformation and the period of the factor transformations (the periods of the latter then being neces-

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† "Infinite Groups Generated by Conformal Transformations of Period 2 (Involutions and Symmetries)," American Journal of Mathematics, XXXVIII, 2, 1916.

sarily equal). In any case, this condition can be obviated by either changing the period of the factor transformations or factoring into three transformations of given period. The complete result is given in Theorem VI.

In the case of reverse conformal transformations, it will be shown that all such transformations are of irreducible period 2; therefore no others exist than those discussed by Kasner.

The parametric form utilized for transformations of regular period is a general solution of Babbage's equation,* and the reduction to the form $f = \varphi^{-1}\epsilon\varphi$ yields $\varphi f = \epsilon\varphi$, a special case of Schröder's equation.†

In the following discussion, the terms function and transformation are used interchangeably, and questions of convergence are not gone into, the analysis being purely formal.

PRELIMINARY DISCUSSION; PARAMETRIC FORM OF PERIODIC TRANSFORMATIONS.

1. Let $f(z)$ be defined by

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{1}$$

where $\lambda_1 \neq 0$. We may introduce the notation

$$f_1(z) = f(z), \qquad f_{s+1}(z) = f[f_s(z)]; \qquad s = 1, 2, 3, \cdots. \tag{2}$$

Assuming the transformation defined by (1) to be of period n, we have

$$f_n(z) = z, \qquad f_{kn+s}(z) = f_s(z); \qquad k = 1, 2, 3, \cdots. \tag{3}$$

We have obviously

$$\lambda_1^n = 1. \tag{4}$$

If in (1) we choose $\lambda_1 = 1$, we have

$$f_n(z) = z + n\lambda_2 z^2 + \cdots;$$

hence we must have $\lambda_2 = 0$. Similarly, all the other coefficients vanish. Hence we have

THEOREM I. *If the transformation*

$$\lambda_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots$$

be of period n, we must have $\lambda_1^n = 1$; if $\lambda_1 = 1$, the transformation reduces to identity.

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* S. Pincherle: "Functional Equations and Operations," *Encyklopädie d. Math. Wiss.*, II, A 11, and *Encyclopedie d. Sci. Math.*, II, 26. Also O. Rausenberger, "Lehrbuch der Theorie der periodischen Funktionen," Leipzig, 1884, p. 162; A. A. Bennett, "The Iteration of Functions of One Variable," *Annals of Math.*, 2d series, 17 (1915).

† Ibid. In our case ϵ is taken so that $\epsilon^n = 1$. G. A. Pfeiffer, *Trans. Am. Math. Soc.*, XVIII, 2, pp. 185–198, considers the complementary case, $|\epsilon| = 1$ with incommensurable angle, and discusses the convergence and divergence of the solutions obtained.

2. Assuming (1) to be of period n, we may write it in the form

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^n = 1. \tag{5}$$

Suppose now ϵ *not* a primitive root of $\epsilon^n = 1$. Suppose ϵ a primitive root of $\epsilon^m = 1$. Then $rm = n$, where $r > 1$. Denoting $f_m(z)$ by $g(z)$, we have

$$g_r(z) = f_{rm}(z) = z.$$

We have obviously $g(z) = \epsilon^m z + \cdots$; hence, by Theorem I,

$$g(z) = f_m(z) = z.$$

Hence the transformation (5) is of period m. We have then

THEOREM II. *If a periodic transformation be expressed in the form*

$$\epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots,$$

where

$$\epsilon^m = 1,$$

*then the transformation is of period m.**

Hence in the form (5) we may conveniently restrict ourselves to the case where ϵ is a primitive root of $\epsilon^n = 1$. In the following discussion, we shall presume this to be the case.

3. Suppose we consider the n variables $z_1, z_2, \cdots, z_n$, between which we have the $n - 1$ relations

$$z_{t+1} = f(z_t); \qquad t = 1, 2, \cdots, n - 1, \tag{6}$$

where

$$f(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \tag{7}$$

ϵ being a primitive nth root of unity. We may introduce the linear substitution

$$z_t = \epsilon^t x_1 + \epsilon^{2t} x_2 + \cdots + \epsilon^{st} x_s + \cdots + \epsilon^{nt} x_n; \quad t = 1, 2, \cdots, n. \tag{8}$$

This is reversible; we have

$$x_s = \frac{1}{n}\left(\epsilon^{-s} z_1 + \epsilon^{-2s} z_2 + \cdots + \epsilon^{-ts} z_t + \cdots + \epsilon^{-ns} z_n\right); \quad s = 1, 2, \cdots, n. \tag{9}$$

By means of (6) and (7), each variable z can be expressed formally as a power series in z_1. We have

$$z_t = \epsilon^{t-1} z_1 + \cdots; \qquad t = 2, 3, \cdots, n. \tag{10}$$

* If (5) take the form

$$f(z) = \epsilon z + \lambda_r z^r + \cdots,$$

we obtain

$$f_n(z) = \epsilon^n z + \epsilon^{n-1} \lambda_r [1 + \epsilon^{r-1} + \epsilon^{2(r-1)} + \cdots + \epsilon^{(n-1)(r-1)}] z^r + \cdots.$$

Hence if $r - 1 \equiv 0 \pmod n$, we must have $\lambda_r = 0$. Hence the next coefficient to appear after the first cannot be of order $kn + 1$.

From (9), then,

$$x_1 = \epsilon^{-1}z_1 + \cdots. \tag{11}$$

From (11), we have z_1 expressible as a power series in x_1. Hence from (10) each z is so expressible. From (9), then, each x can be expanded formally as a power series in x_1 with coefficients uniquely determined in terms of those of f in (7). We may then write

$$x_s = A_{s,\,1}x_1 + A_{s,\,2}x_1^2 + \cdots + A_{s,\,r}x_1^r + \cdots; \quad s = 2, 3, \cdots, n. \tag{12}$$

Hence the $n - 1$ relations (6) among the variables z may be replaced by the $n - 1$ implicit relations (12) where each variable x is expressed in terms of the variables z in the form (9). If z_2 is defined in terms of z_1 in the form (7), then we may introduce the additional variables $z_3, z_4, \cdots, z_n$, similarly defined; hence z_2 may be defined in terms of z_1 in the implicit form (12), involving the elimination of the additional variables $z_3, z_4, \cdots, z_n$.

4. Suppose now the transformation defined by (7) to be of period n. Then, since $z_t = f_{t-1}(z_1)$,

$$z_{n+1} = f_n(z_1) = z_1 = f(z_n). \tag{13}$$

Using the substitution (8), we have then the following n relations between the variables x:

$$\begin{aligned}
&\epsilon^2x_1 + \cdots + \epsilon^{2s}x_s + \cdots + \epsilon^{2n}x_n \\
&\qquad\qquad = f(\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n), \\
&\cdot\quad\cdot\,, \\
&\epsilon^{t+1}x_1 + \cdots + \epsilon^{s(t+1)}x_s + \cdots + \epsilon^{n(t+1)}x_n \\
&\qquad\qquad = f(\epsilon^t x_1 + \cdots + \epsilon^{st}x_s + \cdots + \epsilon^{nt}x_n), \\
&\cdot\quad\cdot\,, \\
&\epsilon^n x_1 + \cdots + \epsilon^{sn}x_s + \cdots + \epsilon^{nn}x_n \\
&\qquad\qquad = f(\epsilon^{n-1}x_1 + \cdots + \epsilon^{s(n-1)}x_s + \cdots + \epsilon^{n(n-1)}x_n), \\
&\epsilon x_1 + \cdots + \epsilon^s x_s + \cdots + \epsilon^n x_n \\
&\qquad\qquad = f(\epsilon^n x_1 + \cdots + \epsilon^{sn}x_s + \cdots + \epsilon^{nn}x_n).
\end{aligned} \tag{14}$$

From the previous discussion, we note that the first $n - 1$ of equations (14) will yield unique expansions in power series of x_1 in the form (12) for the variables $x_2, x_3, \cdots, x_n$. We may represent the relations (12) in the following notation:

$$x_s = \theta_s(x_1); \qquad s = 2, 3, \cdots, n. \tag{15}$$

If we now introduce in (14) the substitution

$$x_s = \epsilon^s y_s; \qquad s = 1, 2, \cdots, n, \tag{16}$$

equations (15) become

$$y_s = \epsilon^{-s}\theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{17}$$

We should observe, however, that the last $n-1$ equations of (14) also yield definite power series in x_1 for the other variables x and that these must be identical with the relations (15). But on introducing the quantities y in the first $n-1$ equations of (14), in the form (16), we observe that the relations between the variables y are identical with those between the corresponding variables x in the last $n-1$ equations of (14). Hence the relations between the variables y must be the same as those between the corresponding variables x. Hence we have

$$y_s = \theta_s(y_1); \qquad s = 2, 3, \cdots, n. \tag{18}$$

These of course must be consistent with (17). Hence

$$\theta_s(y_1) = \epsilon^{-s}\theta_s(\epsilon y_1); \qquad s = 2, 3, \cdots, n. \tag{19}$$

Replacing y_1 in (19) successively by ϵy_1, we obtain, on dropping the subscript,

$$\theta_s(y) = \epsilon^{-ts}\theta_s(\epsilon^t y); \qquad s = 2, 3, \cdots; \qquad t = 1, 2, 3, \cdots. \tag{20}$$

From the form (12), (20) may be written

$$\theta_s(y) = \epsilon^{-ts}\sum_{r=1}^{\infty} \epsilon^{tr}A_{s,\,r}\cdot y^r; \quad s = 2, 3, \cdots, n; \quad t = 1, 2, 3, \cdots. \tag{21}$$

Letting t assume the range of values* 1 to n in (21), and adding the resulting series, we obtain the form

$$\theta_s(y) = A_{s,\,s}y^s + A_{s,\,s+n}y^{s+n} + A_{s,\,s+2n}y^{s+2n} + \cdots; \quad s = 2, 3, \cdots, n. \tag{22}$$

From the form of the series (22) we observe that the double subscripts are not now necessary. We may then write the series (15) in the form

$$\begin{aligned} x_2 &= a_2x_1^2 + a_{n+2}x_1^{n+2} + a_{2n+2}x_1^{2n+2} + \cdots, \\ x_3 &= a_3x_1^3 + a_{n+3}x_1^{n+3} + a_{2n+3}x_1^{2n+3} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots, \\ x_s &= a_sx_1^s + a_{n+s}x_1^{n+s} + a_{2n+s}x_1^{2n+s} + \cdots, \\ &\cdots\cdots\cdots\cdots\cdots, \\ x_n &= a_nx_1^n + a_{2n}x_1^{2n} + a_{3n}x_1^{3n} + \cdots. \end{aligned} \tag{23}$$

Replacing the variables x in (23)† by the corresponding expressions in the variables z from (9), we have then $n-1$ implicit relations among those variables in place of the $n-1$ direct relations (6).

5. If we now write

$$\frac{1}{n}(\epsilon^{-1}z_1 + \epsilon^{-2}z_2 + \cdots + \epsilon^{-t}z_t + \cdots + \epsilon^{-n}z_n) = r, \tag{24}$$

* This is equivalent to permuting the substitution (16) in equations (14).

† These equations have already been obtained by Bennett, loc. cit.

equations (23) may be written

$$\begin{aligned}
&\frac{1}{n}(\epsilon^{-2}z_1+\epsilon^{-4}z_2+\cdots+\epsilon^{-2t}z_t+\cdots+\epsilon^{-2n}z_n)=a_2r^2+a_{n+2}r^{n+2}+\cdots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\frac{1}{n}(\epsilon^{-s}z_1+\epsilon^{-2s}z_2+\cdots+\epsilon^{-ts}z_t+\cdots+\epsilon^{-ns}z_n)=a_sr^s+a_{n+s}r^{n+s}+\cdots, \qquad (25)\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&\frac{1}{n}(\epsilon^{-n}z_1+\epsilon^{-2n}z_2+\cdots+\epsilon^{-tn}z_t+\cdots+\epsilon^{-nn}z_n)=a_nr^n+a_{2n}r^{2n}+\cdots.
\end{aligned}$$

Multiplying equations (24) and (25) seriatim by ϵ^t, ϵ^{2t}, $\cdots$, ϵ^{st}, $\cdots$, ϵ^{nt}, and adding, we obtain

$$\begin{aligned}
z_t = \epsilon^t r &+ \epsilon^{2t}(a_2r^2+a_{n+2}r^{n+2}+\cdots)\\
&+\cdots\cdots\cdots\\
&+\epsilon^{st}(a_sr^s+a_{n+s}r^{n+s}+\cdots) \qquad (26)\\
&+\cdots\cdots\cdots\\
&+\epsilon^{nt}(a_nr^n+a_{2n}r^{2n}+\cdots),
\end{aligned}$$

for all values of t from 1 to n. If we now define the function φ by

$$\varphi(z) = z + \sum_{k=0}^{\infty}\sum_{r=1}^{n-1} a_{kn+r+1}z^{kn+r+1}, \tag{27}$$

equations (26) take the form

$$z_t = \varphi(\epsilon^t r); \qquad t = 1, 2, \cdots, n. \tag{28}$$

In particular,

$$z_1 = \varphi(\epsilon r), \qquad z_2 = \varphi(\epsilon^2 r), \tag{29}$$

which are equivalent to

$$z_1 = \varphi(r), \qquad z_2 = \varphi(\epsilon r), \tag{29$'$}$$

defining z_2 as a function of z_1 in terms of the parameter r. Furthermore, every pair of equations (28) determines either z uniquely as an integral power series in the other with coefficients expressible as polynomials in the coefficients a in (27) of the same and lower orders. This is obviously also true if instead of φ as defined by (27) we use the more general form

$$\psi(r) = r + \sum_{t=2}^{\infty} a_t r^t. \tag{30}$$

6. From the form of equations (28) we observe that if for an arbitrary set of coefficients a we obtain $z_2 = f(z_1)$, we must also have $z_3 = f(z_2)$, etc. Hence $z_{n+1} = f(z_n)$. But $z_1 = z_{n+1} = z_{2n+1}$, etc. Hence $z_1 = f(z_n)$, from which $f_n(z) = z$. Hence $f(z)$ is of period n. Furthermore, if z_α and z_β be any two of the z's defined by (28), either is expressible as a periodic

function of the other. If $|\alpha - \beta|$ is prime to n, the order is also n. If only equations (29) are given, or the corresponding equations with ψ instead of φ, the remaining z's may be introduced in the form (28), yielding z_2 as a function of z_1 of period n. Hence we have

THEOREM III. *Every transformation of period n, $y = f(z)$, can be put uniquely into the form*

$$z = \varphi(r), \qquad y = \varphi(\epsilon r); \qquad \epsilon^n = 1,$$

where

$$\varphi(r) = r + \sum_{k=0}^{\infty} \sum_{t=1}^{n-1} a_{kn+t+1} r^{kn+t+1};$$

and every solution of these equations for arbitrary values of the coefficients defines $y = f(z)$ uniquely as a transformation of period n.

*In other words, a necessary and sufficient condition for $y = f(z)$ to be a transformation of period n is that it be a solution of equations of the form $z = \varphi(r)$, $y = \varphi(\epsilon r)$.**

Hence the two forms are equivalent, and we may conveniently confine our attention to the inclusive form (29′).

7. If the transformation (5) result from the elimination of r between the equations (29′), then we obtain

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } \lambda\text{'s and } a\text{'s of order} < t),$$

which can obviously be put into the form

$$\lambda_t = (\epsilon^t - \epsilon)a_t + (\text{polynomial in } a\text{'s of order} < t). \tag{31}$$

The quantities a in (31) are arbitrary. For each new λ_t, where t is not of the form $kn + 1$, a new arbitrary a_t is introduced. Furthermore, each a can be expressed as a polynomial in λ's of corresponding and lower orders. Hence the λ's in (5) of orders not of the form $kn + 1$ can be taken arbitrarily, the remaining λ's being then determined necessarily in the form

$$\lambda_{kn+1} = R_k(\lambda_{kn}, \lambda_{kn-1}, \cdots); \qquad k = 1, 2, \cdots, \tag{32}$$

where the R's are rational integral functions. Hence we have

THEOREM IV. *If*

$$y = \epsilon z + \lambda_2 z^2 + \cdots; \qquad \epsilon^n = 1$$

be a transformation of period n, all coefficients of orders not of the form $kn + 1$

* If $n = 2$, $\epsilon = -1$, and equations (27) and (29′) yield $z_1 = r + a_2r^2 + a_4r^4 + \cdots$, $z_2 = -r + a_2r^2 + a_4r^4 + \cdots$, from which

$$\frac{z_2 + z_1}{2} = a_2\left(\frac{z_2 - z_1}{2}\right)^2 + a_4\left(\frac{z_2 - z_1}{2}\right)^4 + \cdots,$$

the form utilized by Kasner.

may be taken arbitrarily, the remaining coefficients being determined as necessary rational integral functions of the coefficients of lower order.

In other words, $k(n-1)$ out of the kn coefficients following the first may be taken arbitrarily.

The relations (32) are then necessary and sufficient for the transformation (5) to be of period n. If now, instead of (29′), we define the relation $y = f(z)$ in the form

$$z = \psi(r), \qquad y = \psi(\epsilon r), \tag{33}$$

where ψ has the form (30), then f is also of period n. In other words, (33) is no more general than (29′). In this case too the relations (31) will not yield expressions in λ's for the a's of order $kn+1$. On the other hand, the λ's of order $kn+1$ will be expressed in terms of a's of order $k'n+1$, $k' < k$, as well as the other a's. But since the relations (32) between the λ's are *necessary*, it follows that the a's of order $k'n+1$ will be automatically eliminated on eliminating the remaining a's from the expressions for λ_{kn+1} and the λ's of lower order.

Hence, as already observed, any two equations $z_\alpha = \varphi(\epsilon^\alpha r)$ and $z_\beta = \varphi(\epsilon^\beta r)$, of (28), or the corresponding equations with ψ instead of φ, determine either z as a periodic function of the other. The period is the quotient between n and the greatest common divisor of n and $|\alpha - \beta|$.

8. If the function

$$f(z) = \lambda_1 z + \lambda_2 z^2 + \cdots \tag{34}$$

result from the relation

$$f(z) = g^{-1}[\epsilon g(z)]; \qquad \epsilon^n = 1, \tag{35}$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0, \tag{36}$$

then $f(z)$ is obviously of period n. We shall now show that every periodic transformation $f(z)$ can be expressed in the form (35).*

Suppose f is given where, symbolically, $f^n = 1$. We inquire now whether g can always be found so as to satisfy (35). This condition is equivalent to

$$g[f(z)] = \epsilon g(z). \tag{37}$$

It is to be observed that, whether we obtain f when g is given, or g when f is given, the *same* set of formal identities obtained by equating the coefficients of like powers of z in (37) must be considered. These take the form

$$\begin{aligned} &A_1\lambda_1 = \epsilon A_1, \qquad A_1\lambda_2 + A_2\epsilon^2 = \epsilon A_2, \qquad \cdots, \\ &A_1\lambda_t + 2A_2\epsilon\lambda_{t-1} + \cdots + A_t\epsilon^t = \epsilon A_t, \qquad \cdots. \end{aligned} \tag{38}$$

We observe that A_1 may be taken arbitrarily subject, of course, to the

* Cf. S. Pincherle, O. Rausenberger and A. A. Bennett, loc. cit.

condition $A_1 \neq 0$. We have $\lambda_1 = \epsilon$. Furthermore, each coefficient A_t, $t < n+1$, is expressed as A_1 (polynomial in λ's). Hence we have λ_{n+1} expressed as a necessary rational integral function of λ's of lower order. Similarly, λ's of order $kn+1$ are expressed as rational integral functions of lower λ's and of arbitrary A's of order $k'n+1$, $k' < k$. But since every solution $f(z)$ where $g(z)$ in (35) is given is of period n, we have then a set of *necessary* conditions for the coefficients λ of order $kn+1$ of the form (32). Hence on eliminating the coefficients A of orders other than those of the form $kn+1$ from the expressions for λ_{kn+1} and λ's of lower order, the coefficients $A_{k'n+1}$ ($k' < k$) must be eliminated automatically, the resulting conditions being identical with those represented by (32).

Furthermore, if (34) is given, we can find a function $g(z)$ consistent with (35) only if the same set of conditions (32) are satisfied between the coefficients λ. But as just seen these conditions are none other than those necessary for $f(z)$ to be a periodic transformation of order n. Hence every transformation (34) of order n can be expressed in the form (35). Furthermore, given g, f results uniquely, but not vice versa, since the coefficients of form A_{kn+1} may be taken arbitrarily, subject to the single restriction $A_1 \neq 0$. Hence we may conveniently choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

from which g takes the form

$$g(z) = z + \sum_{k=0}^{\infty} \sum_{r=1}^{n-1} A_{kn+r+1} z^{kn+r+1}, \tag{39}$$

in which case, from the form of equations (38), each coefficient A is determined uniquely in terms of the coefficients λ of the given transformation (34). Hence we have

THEOREM V. *Every transformation of the form*

$$f(z) = g^{-1}[\epsilon g(z)], \qquad \epsilon^n = 1,$$

where

$$g(z) = A_1 z + A_2 z^2 + \cdots; \qquad A_1 \neq 0,$$

is of period n, and every transformation $f(z)$ of period n may be expressed in the form $g^{-1}[\epsilon g(z)]$; furthermore, if we choose

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, \cdots,$$

the function $g(z)$ corresponds uniquely to $f(z)$.

In other words, every periodic transformation is conformally reducible to the form $Y = \epsilon Z$, a rotation about the origin through the angle $2\pi/n$.*

It is interesting to compare the functions φ and g in (27) and (39). Obviously each corresponds uniquely to the other and either may be used in place of the other or its inverse in the above equations.

* A simple example of a periodic transformation is obtained by taking $g(z) = z/(z-1)$, from which $f(z) = \epsilon z/[(\epsilon - 1)z + 1]$.

9. The method followed in the preceding discussion will enable us to write down readily the general solutions of some special additional functional equations. Suppose f given, where $f^n = 1$, and it is required to find g where, symbolically, $g^m = f$.* We may anticipate the number of arbitrary coefficients involved in the general solution. In that for $g^{mn} = 1$, out of tmn coefficients following the first a proportion of $(mn - 1)/mn$ may be taken arbitrary. For $f^n = 1$, the proportion $(n - 1)/n$ are arbitrary. Having assigned f, the proportion still remaining arbitrary is

$$\frac{mn-1}{mn} - \frac{n-1}{n} = \frac{m-1}{mn}.$$

Suppose $f = \varphi^{-1}\epsilon\varphi$, where $\epsilon^n = 1$, and suppose h the general solution of $h^m = \epsilon$. Then $g = \varphi^{-1}h\varphi$ is the general solution of $g^m = f$. We have then to consider the functional equation $h^m = \epsilon$.

Let $\omega^m = \epsilon$; then $\omega^{mn} = 1$. Putting mn for n and ω for ϵ in (3) we observe that $z_{km+s} = \epsilon^k z_s$ for all values of k and s. Equation (24) becomes

$$\frac{1}{m}(\omega^{-1}z_1 + \omega^{-2}z_2 + \cdots + \omega^{-m}z_n) = r.$$

For all values of $s \not\equiv 1 \pmod{n}$, we have from (25)

$$\frac{1}{m}(\omega^{-s}z_1 + \omega^{-2s}z_2 + \cdots + \omega^{-ms}z_m) = a_s r^s + a_{n+s}r^{n+s} + \cdots;$$

for all other values of s the left members of (25) vanish. Applying the method of Section 5, we have

$$z_t = \varphi_1(\omega^t r),$$

where

$$\varphi_1(r) = r + \sum_{p=0}^{\infty}\sum_{k=1}^{m-1} a_{(pm+k)n+1}r^{(pm+k)n+1}. \tag{40}$$

Hence h is defined by

$$z = \varphi_1(r), \qquad h(z) = \varphi_1(\omega r), \qquad \omega^m = \epsilon, \qquad \epsilon^n = 1.$$

Hence the most general solution for $g^m = f$ where $f^n = 1$ is given by

$$g = \varphi^{-1}h\varphi = \varphi^{-1}\varphi_1\omega\varphi_1^{-1}\varphi.\dagger$$

* This is a very special case of the more general problem, the solution of which is equivalent to finding an n-section of a curvilinear angle in the sense employed by Kasner as an extension of the idea of symmetry and corresponding bisection due to Schwarz. See G. A. Pfeiffer, "On the Conformal Geometry of Analytic Arcs," AMER. JOUR. MATH., XXXVII, 4 (1915).

† A less specific form may be obtained more directly. Suppose g in form $\lambda^{-1}\omega\lambda$. Then $\lambda^{-1}\epsilon\lambda = f = \varphi^{-1}\epsilon\varphi$, whence $\varphi\lambda^{-1}\epsilon = \epsilon\varphi\lambda^{-1}$. Putting μ for $\varphi\lambda^{-1}$ we have $\mu(\epsilon z) = \epsilon\mu(z)$; hence μ must be of the form $\mu(z) = z\theta(z^n)$. Hence if φ is given we may choose $\lambda = \mu^{-1}\varphi$ in $g = \lambda^{-1}\omega\lambda = \varphi^{-1}\mu\omega\mu^{-1}\varphi$ as the most general solution of $g^m = f$. By Theorem V it is evident that there is a sufficient number of arbitrary constants, though not indicated in the very explicit form (40).

We may similarly write down the general solution of the functional equation

$$z + f(z) + f_2(z) + \cdots + f_{n-1}(z) = 0.$$

We observe that if $1 + f + \cdots + f^{n-1} = 0$, then $f^n - 1 = 0$ and f is a periodic function. The converse is not necessarily the case, as is readily verified in the special case

$$f(z) = \epsilon z + \epsilon z^2 + \epsilon z^3 + \cdots; \qquad \epsilon^3 = 1,$$

which satisfies $f^3 = 1$ but not $1 + f + f^2 = 0$.

We note that the left member of the last of equations (25) vanishes according to this condition. The general solution is then at once written in the form

$$z = \varphi_2(r), \qquad f(z) = \varphi_2(\epsilon r),$$

where

$$\varphi_2(z) = z + \sum_{k=0}^{\infty} \sum_{s=1}^{n-2} a_{kn+s+1} z^{kn+s+1}. \tag{41}$$

Similarly, the general solutions are obtained for equations formed by equating any of the left members of (25) to zero. A similar result may be obtained when the condition (yielding $n - 1$ similar conditions) is a linear relation obtained by eliminating $s - 1$ z's after equating s of these expressions to zero.

It may be shown that, corresponding to every periodic function f, where $f_n = 1$, there are others F, for which, symbolically,

$$1 + F + F^2 + \cdots + F^{n-1} = 0.*$$

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* Instead of the substitution (8), we may effect the substitution $z_k = \sum_{t=1}^{n} \alpha_{k,\,t} u_t$ ($k = 1, 2, \cdots, n$), where the α's form a set of orthogonal numbers. We shall for convenience write $\alpha_{p,\,n+q} = \alpha_{p+n,\,q} = \alpha_{p,\,q}$, and choose $\alpha_{k,\,n} = 1/\sqrt{n}$ ($k = 1, 2, \cdots, n$). The last requires $\sum_{s=1}^{n} \alpha_{s,\,r} = 0$ ($r = 1, 2, \cdots, n - 1$).

We have then n equations in n u's. Each $n - 1$ of these determine all u's in terms of u_1: $u_k = \theta_k(u_1)$, $k = 2, 3, \cdots, n$. Further, the functions so determined are unique. We now introduce the further substitution $u_k = \sum_{t=1}^{n-1} \beta_{k,\,t} v_t$, $u_n = v_n$ ($k = 1, 2, \cdots, n - 1$), where $\beta_{s,\,t} = \sum_{p=1}^{n} \alpha_{p,\,s}\alpha_{p+1,\,t}$. It is readily verified that the $(n - 1)^2$ quantities β form a set of orthogonal numbers.

We now have

$$\begin{aligned} z_k &= \sum_{t=1}^{n} \alpha_{k,\,t} u_t = \alpha_{k,\,n} v_n + \sum_{t=1}^{n-1} \left(\alpha_{k,\,t} \sum_{s=1}^{n-1} \beta_{t,\,s} v_s \right) \\ &= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left(v_s \sum_{t=1}^{n-1} \alpha_{k,\,t} \beta_{t,\,s} \right) \\ &= \alpha_{k+1,\,n} v_n + \sum_{s=1}^{n-1} \left[v_s \sum_{p=1}^{n} \left(\alpha_{p+1,\,s} \sum_{t=1}^{n-1} \alpha_{k,\,t} \alpha_{p,\,t} \right) \right] \\ &= \sum_{s=1}^{n-1} \left[v_s \left(-\frac{1}{n} \sum_{p=1}^{n} \alpha_{p+1,\,s} + \alpha_{k+1,\,s} \right) \right] + \alpha_{k+1,\,n} v_n \\ &= \sum_{s=1}^{n} \alpha_{k+1,\,s} v_s. \end{aligned}$$

FACTORIZATION; GROUPS OF PERIODIC TRANSFORMATIONS.

10. Kasner has considered the question whether any given transformation may be factored into two transformations each of period 2. In other

Hence z_k is the same function of the v's as z_{k+1} is of the corresponding u's. But the latter are uniquely determined by any set of $n-1$ equations. Hence the relations among the v's are the same as those among the corresponding u's. Hence the θ-functions remain unchanged under transformations of the form $u_k \to \sum_{t=1}^{n-1} \beta_{k,\,t} u_t,\ u_n \to u_n\ (k = 1, 2, \cdots, n-1)$.

Putting $F(u_1) = \sum_{k=1}^{n-1} \beta_{1,\,k}\theta_k(u_1)$, we have $\theta_k[F(u_1)] = \sum_{p=1}^{n-1} \beta_{k,\,p}\theta_p(u_1)$. We have then

$$\begin{aligned}
F_2(u_1) &= \sum_{k=1}^{n-1}\left[\beta_{1,\,k}\sum_{p=1}^{n-1}\beta_{k,\,p}\theta_p(u_1)\right] = \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\beta_{1,\,k}\beta_{k,\,p}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{k=1}^{n-1}\left\{\left(\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+1,\,k}\right)\left(\sum_{t=1}^{n}\alpha_{t,\,k}\alpha_{t+1,\,p}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s,t=1}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\alpha_{s,\,1}\alpha_{s+2,\,p}\sum_{k=1}^{n-1}\alpha^2_{s+1,\,k}\right)+\sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\left(\alpha_{s,\,1}\alpha_{t+1,\,p}\sum_{k=1}^{n-1}\alpha_{s+1,\,k}\alpha_{t,\,k}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left(\frac{n-1}{n}\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}-\frac{1}{n}\sum_{\substack{s,t=1\\ s+1\neq t}}^{n}\alpha_{s,\,1}\alpha_{t+1,\,p}\right)\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}-\frac{1}{n}\left(\sum_{s=1}^{n}\alpha_{s,\,1}\right)\left(\sum_{t=1}^{n}\alpha_{t+1,\,p}\right)\right\}\right]\\
&= \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n}\alpha_{s,\,1}\alpha_{s+2,\,p}\right] = \sum_{p=1}^{n-1} a_{p,2}\,\theta_p(u_1)
\end{aligned}$$

if we introduce $a_{p,\,q} = \sum_{s=1}^{n} \alpha_{s,\,1}\alpha_{s+q,\,p}$.

Similarly, we obtain

$$F_m(u_1) = \sum_{p=1}^{n-1} a_{p,\,m}\theta_p(u_1).$$

We have then

$$\begin{aligned}
u_1 + \sum_{p=1}^{n-1} F_p(u_1) &= u_1 + \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1} a_{p,s}\right] = u_1 + \sum_{p=1}^{n-1}\left[\theta_p(u_1)\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right)\right]\\
&= u_1 + u_1\sum_{s=1}^{n-1}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,1}\right)\\
&\qquad + \sum_{p=2}^{n-1}\left[\theta_p(u_1)\left\{\sum_{s=1}^{n}\left(\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t+s,\,p}\right)-\sum_{t=1}^{n}\alpha_{t,\,1}\alpha_{t,\,p}\right\}\right]\\
&= u_1 - u_1\sum_{t=1}^{n}\alpha^2_{t,\,1} = 0.
\end{aligned}$$

The last result also follows from the relation $F_m(u_1) = \sum_{t=1}^{n} \alpha_{t,\,1} z_{t+m}$, readily deduced. In a similar way, we may also obtain

$$\theta_k^{(m)}(u_1) = \sum_{t=1}^{n-1}\left(u_t\sum_{p=1}^{n}\alpha_{p,\,k}\alpha_{p+m,\,t}\right),$$

where

$$\theta_k^{(m)}(u_1) = \theta_k^{(m-1)}[F(u_1)], \qquad \theta'_k(u_1) = \theta_k(u_1),$$

yielding

$$\sum_{m=1}^{n}\theta_k^{(m)}(u_1) = 0,$$

from which the previous result also follows.

words, given F, can f and g be found such that, symbolically,

$$F = gf; \qquad f^2 = 1, \qquad g^2 = 1?$$

We may generalize this inquiry as follows: Given $F(z)$, can $f(z)$ and $g(z)$ be obtained where, with the notation (2),

$$F(z) = g[f(z)]; \qquad f_m(z) = z, \qquad g_n(z) = z? \tag{42}$$

We shall assume these functions to be defined as follows:

$$\begin{aligned} F(z) &= K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \\ f(z) &= \epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots, \qquad \epsilon_1^m = 1; \\ g(z) &= \epsilon_2 z + \mu_2 z^2 + \mu_3 z^3 + \cdots, \qquad \epsilon_2^n = 1. \end{aligned} \tag{43}$$

We shall assume here that ϵ_1 and ϵ_2 are primitive roots of unity of orders m and n, respectively. It is to be observed that all the coefficients λ and μ in (43) may be taken arbitrarily, consistent with conditions (32). For our purpose, we shall inquire more minutely as to the form of these conditions.

11. From (29′) and (27) we have the identity

$$\begin{aligned} &\epsilon_1 r + (\epsilon_1^2 a_2 r^2 + \cdots + \epsilon_1^m a_m r^m) + (\epsilon_1^2 a_{m+2} r^{m+2} + \cdots + \epsilon_1^m a_{2m} r^{2m}) + \cdots \\ &\quad = \epsilon_1[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots] \\ &\quad + \lambda_2[r + (a_2 r^2 + \cdots + a_m r^m) + (a_{m+2} r^{m+2} + \cdots + a_{2m} r^{2m}) + \cdots]^2 \\ &\quad + \lambda_3 [\qquad\qquad\qquad]^3 + \cdots. \end{aligned} \tag{44}$$

On equating coefficients these yield

$$\begin{aligned} &\lambda_2 = (\epsilon_1^2 - \epsilon_1) a_2, \qquad \lambda_3 = (\epsilon_1^3 - \epsilon_1) a_3 - 2(\epsilon_1^2 - \epsilon_1) a_2^2, \qquad \text{etc.}; \\ &a_2 = \frac{\lambda_2}{\epsilon_1^2 - \epsilon_1}, \qquad a_3 = \frac{\lambda_3}{\epsilon_1^3 - \epsilon_1} + \frac{2\lambda_2^2}{(\epsilon_1^3 - \epsilon_1)(\epsilon_1^2 - \epsilon_1)}, \quad \text{etc.} \end{aligned} \tag{45}$$

Similarly, we obtain corresponding relations for the coefficients μ. Equating the coefficients of z^{pm+1} in (44), and using the set (45), we readily obtain for (32)

$$\begin{aligned} \lambda_{pm+1} &= \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1} \lambda_2 \lambda_{pm} + \frac{3\epsilon_1^2 - (pm-1)}{\epsilon_1^3 - \epsilon_1} \lambda_3 \lambda_{pm-1} \\ &+ \frac{\begin{array}{c} 2(\epsilon_1^2 + \epsilon_1 - 1)(pm - 1) - 2\epsilon_1^2(2\epsilon_1 - 1) \\ - \frac{1}{2}(\epsilon_1 + 1)(pm - 1)(pm - 2) \end{array}}{(\epsilon_1 + 1)(\epsilon_1^2 - \epsilon_1)^2} \lambda_2^2 \lambda_{pm-1} \\ &+ (\text{lower orders of } \lambda), \end{aligned} \tag{46}$$

and similar relations for μ_{sn+1}. The form (46) presupposes $m > 2$.

12. The condition $F = fg$ yields the identity

$$\begin{aligned} K_1 z + K_2 z^2 + K_3 z^3 + \cdots = \quad & \epsilon_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots) \\ & + \mu_2(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^2 \\ & + \mu_3(\epsilon_1 z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots)^3 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot . \end{aligned} \tag{47}$$

If, consistent with conditions (46), and the corresponding ones for the coefficients μ, the λ's and μ's can be chosen so as to satisfy (47) identically, then $F(z)$ can be factored into two periodic transformations of orders m and n respectively. From (47), we have as a necessary condition

$$K_1 = \epsilon_1 \epsilon_2. \tag{48}$$

Hence K_1 must be a root of unity. Equating coefficients of higher powers,

$$K_2 = \epsilon_2 \lambda_2 + \epsilon_1^2 \mu_2, \tag{49}$$

$$K_3 = \epsilon_2 \lambda_3 + 2\epsilon_1 \lambda_2 \mu_2 + \epsilon_1^3 \mu_3, \tag{50}$$

$$K_4 = \epsilon_2 \lambda_4 + (2\epsilon_1 \lambda_3 + \lambda_2^2)\mu_2 + 3\epsilon_1^2 \lambda_2 \mu_3 + \epsilon_1^4 \mu_4, \tag{51}$$

$$K_5 = \epsilon_2 \lambda_5 + (2\epsilon_1 \lambda_4 + 2\lambda_2 \lambda_3)\mu_2 + (3\epsilon_1^2 \lambda_3 + 3\epsilon_1 \lambda_2^2)\mu_3 + 4\epsilon_1^3 \lambda_2 \mu_4 + \epsilon_1^5 \mu_5, \tag{52}$$

etc. The question now is, assuming (48) to be satisfied, can the λ's and μ's always be chosen without involving necessary conditions among the K's or the indices m and n? Equating the coefficients of z^t, z^{t-1}, z^{t-2}, we obtain

$$\begin{aligned} K_t \; = & \; [\epsilon_2 \lambda_t + \epsilon_1^t \mu_t] + [2\epsilon_1 \mu_2 \lambda_{t-1} + \epsilon_1^{t-2}(t-1)\lambda_2 \mu_{t-1}] \\ & + [2\lambda_2 \mu_2 + 3\epsilon_1^2 \mu_3]\lambda_{t-2} \\ & + \left[\epsilon_1^{t-3}(t-2)\lambda_3 + \epsilon_1^{t-4}\frac{(t-2)(t-3)}{2}\lambda_2^2\right]\mu_{t-2} \\ & + [\text{lower orders of } \lambda \text{ and } \mu], \end{aligned} \tag{53}$$

$$\begin{aligned} K_{t-1} = & \; [\epsilon_2 \lambda_{t-1} + \epsilon_1^{t-1}\mu_{t-1}] + [2\epsilon_1 \mu_2 \lambda_{t-2} + \epsilon_1^{t-3}(t-2)\lambda_2 \mu_{t-2}] \\ & + [\text{lower orders of } \lambda \text{ and } \mu], \end{aligned} \tag{54}$$

$$K_{t-2} = [\epsilon_2 \lambda_{t-2} + \epsilon_1^{t-2}\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{55}$$

The entire set of conditions obtained are necessary and sufficient for factorization. We observe that for each coefficient K_t in these equations *two* new quantities λ_t and μ_t are introduced, which may be taken arbitrarily except for periodic orders of these coefficients. Further, the new quantities introduced appear with non-zero coefficients. If either λ_t or μ_t is independent, the condition involving K_t can then be satisfied. Hence, the only way for a necessary condition to obtain among the coefficients K is for *both*

the coefficients λ_t and μ_t to be conditioned in terms of coefficients of lower order in the form (46). Hence t must be of the forms both $pm + 1$ and $sn + 1$. We may write

$$t = pm + 1 = sn + 1. \tag{56}$$

Hence λ_t and μ_t in (53) are not independent. These are then to be replaced by the equivalent expressions in terms of λ's and μ's of lower order, from the form (46). Hence a necessary condition for obtaining a necessary condition among the coefficients K is that it be possible to eliminate simultaneously the coefficients λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1} in terms of these coefficients and those of lower order. Using (56), the condition for this is

$$\frac{\epsilon_2[2\epsilon_1 - (t-1)]}{\epsilon_1^2 - \epsilon_1}\lambda_2 + 2\epsilon_1\mu_2 = \epsilon_2\left\{\frac{\epsilon_1[2\epsilon_2 - (t-1)]}{\epsilon_2^2 - \epsilon_2}\mu_2 + \epsilon_1^{-1}(t-1)\lambda_2\right\},$$

which may be written

$$(t-3)\left[\frac{\lambda_2}{\epsilon_1^2 - \epsilon_1} - \frac{\mu_2}{\epsilon_2^2 - \epsilon_2}\right] = 0. \tag{57}$$

The case $t = 3$ requires $m = n = 2$, already considered by Kasner.* We may then confine our attention to the condition

$$(\epsilon_2^2 - \epsilon_2)\lambda_2 - (\epsilon_1^2 - \epsilon_1)\mu_2 = 0, \tag{58}$$

and choose $t > 3$. If λ_2 and μ_2 can be so chosen as *not* to satisfy (58), then no necessary condition is required among the K's. The only condition upon these is given by (49). Eliminating μ_2, we have

$$K_2 = \frac{\epsilon_1\epsilon_2}{\epsilon_1^2 - \epsilon_1}(\epsilon_1\epsilon_2 - 1)\lambda_2. \tag{59}$$

Here K_2 is assigned and λ_2 completely arbitrary. Hence we may choose λ_2 so that (59) is not satisfied unless

$$K_2 = 0 \qquad \text{and} \qquad \epsilon_1\epsilon_2 - 1 = 0. \tag{60}$$

The latter condition requires $K_1 = 1$. Hence we have the result

If $K_1 \neq 1$, the transformation $F(z) = K_1 z + \cdots$ can always be factored into two transformations of order m and n, the orders being so chosen that $K_1 = \epsilon_1\epsilon_2$, where ϵ_1 and ϵ_2 are primitive roots of unity of order m and n respectively.

13. Suppose now that $K_1 = \epsilon_1\epsilon_2 = 1$. Then we must have

$$m = n, \tag{61}$$

* In this case, $\epsilon_1 = \epsilon_2 = -1$, (45) yield $\lambda_3 = -\lambda_2^2$, $\mu_3 = -\mu_2$, and (49) and (50) give $K_3 - K_2 = 0$ as a necessary condition, (48) requiring $K_1 = 1$. Cf. Kasner, loc. cit.

since ϵ_1 and ϵ_2 are primitive roots. Taking further $K_2 = 0$, we have

$$\mu_2 = -\epsilon_2^3\lambda_2. \tag{62}$$

If we now write (46) and the corresponding expression in μ's in the form

$$\begin{aligned}\lambda_{pm+1} &= \frac{2\epsilon_1 - pm}{\epsilon_1^2 - \epsilon_1}\lambda_2\lambda_{pm} + A\lambda_3\lambda_{pm-1} + B\lambda_2^2\lambda_{pm-1} \\ &\qquad + [\text{lower orders of } \lambda], \qquad (63)\\ \mu_{sn+1} &= \frac{2\epsilon_2 - sn}{\epsilon_2^2 - \epsilon_2}\mu_2\mu_{sn} + C\mu_3\mu_{sn-1} + D\mu_2^2\mu_{sn-1} \\ &\qquad + [\text{lower orders of } \mu],\end{aligned}$$

and eliminate λ_{t-1} and μ_{t-1} from equations (53) and (54), we have, on using (56), (60), (61) and (62),

$$\begin{aligned}K_t &+ \frac{t-3}{\epsilon_1^2 - \epsilon_1}\lambda_2 K_{t-1} \\ &= \left[\epsilon_2 A\lambda_3 + \epsilon_2 B\lambda_2^2 - 2\epsilon_2^3\lambda_2^2 + 3\epsilon_1^2\mu_3 - \frac{2\epsilon_2^2(t-3)}{\epsilon_1^2 - \epsilon_1}\lambda_2^2\right]\lambda_{t-2} \\ &\quad + \Bigg[\epsilon_1 C\mu_3 + \epsilon_2^5 D\lambda_2^2 + \epsilon_2^2(t-2)\lambda_3 \\ &\qquad + \epsilon_2^3\frac{(t-2)(t-3)}{2}\lambda_2^2 + \frac{\epsilon_1^2(t-2)(t-3)}{\epsilon_1^2 - \epsilon_1}\Bigg]\mu_{t-2} \\ &\quad + [\text{lower orders of } \lambda \text{ and } \mu].\end{aligned} \tag{64}$$

Equation (55) becomes

$$K_{t-2} = [\epsilon_2\lambda_{t-2} + \epsilon_2\mu_{t-2}] + [\text{lower orders of } \lambda \text{ and } \mu]. \tag{65}$$

The form of (64) presupposes $t > 5$. For the moment we shall assume this satisfied. Here λ_{t-2} and μ_{t-2} in (64) and (65) can be so chosen as to satisfy both, unless the determinant of these coefficients vanishes. The condition for this becomes

$$[\epsilon_2 A - \epsilon_2^2(t-2)]\lambda_3 + [3\epsilon_1^2 - \epsilon_1 C]\mu_3 + \frac{2\epsilon_2(t-5)}{\epsilon_1^2 - 1}\lambda_2^2 = 0.$$

Here λ_2 is independent and the only condition upon λ_3 and μ_3 is (50) Eliminating μ_3 we obtain

$$L\lambda_3 + M\lambda_2^2 + NK_3 = 0, \tag{66}$$

where L, M and N are rational expressions in t and ϵ_2. We have

$$N = \epsilon_2\frac{t-5}{\epsilon_2^2 - 1} \neq 0.$$

In (66), λ_2 and λ_3 are completely independent. Hence these may be chosen

so as not to satisfy (66), unless

$$L = M = 0 \tag{67}$$

identically, and $K_3 = 0$. On substituting their values for A and C, we find these conditions fulfilled. Hence if $K_1 = 1$, and $K_2 = K_3 = 0$, we must resort to further analysis.

14. For this purpose we return to the identity (44). If t be *even*, we may write the general condition in the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} + \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{t/2} M_{(t/2)+1} + \lambda_{(t/2)+1} M_{t/2}, \end{aligned} \tag{68}$$

where

$$M_{t-k} = (k+1)a_{t-k} + (\text{polynomial in lower orders of } a). \tag{69}$$

If t be *odd*, we have the form

$$\begin{aligned} \epsilon_1^t a_t = {} & \epsilon_1 a_t + \lambda_t \\ & + \lambda_2 M_{t-1} + \lambda_{t-1} M_2 \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{k+1} M_{t-k} + \lambda_{t-k} M_{k+1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{(t-1)/2} M_{(t+3)/2} + \lambda_{(t+3)/2} M_{(t-1)/2} \\ & + \lambda_{(t+1)/2} M_{(t+1)/2}. \end{aligned} \tag{70}$$

From (45), we have a_2 and a_3 expressed in terms of the corresponding λ's. Substituting in (69) for $t = 4$, we have a_4 in terms of λ's. Using (68) and (70), we finally have all a's that appear in (44) expressed in the forms (68) and (70) where the M's are expressed in λ's in the form

$$\begin{aligned} M_{t-k} &= \frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} \lambda_{t-k} + (\text{polynomial in lower orders of } \lambda), \\ M_{k+1} &= \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \lambda_{k+1} + (\text{polynomial in lower orders of } \lambda). \end{aligned} \tag{71}$$

If $t - k \equiv 1 \pmod{m}$, the first term indicated in (71) is of course absent. The highest order of λ that multiplies λ_{t-k} is λ_{k+1}, and occurs only in the products $\lambda_{k+1} M_{t-k}$ and $\lambda_{t-k} M_{k+1}$. If $t - k > (t+1)/2$, then λ_{t-k} occurs only to the first power in (68) or (70). The coefficient of λ_{t-k} is

$$\left(\frac{k+1}{\epsilon_1^{t-k} - \epsilon_1} + \frac{t-k}{\epsilon_1^{k+1} - \epsilon_1} \right) \lambda_{k+1} + (\text{polynomial in lower orders of } \lambda).$$

If now we take $t-1$ a multiple of m, we obtain from (68), if t be even,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k+1}\lambda_{t-k} + \cdots + N_{t/2}\lambda_{(t/2)+1} \\ + \text{(polynomial in lower orders of } \lambda), \tag{72}$$

and from (70), if t be odd,

$$\lambda_t = N_2\lambda_{t-1} + N_3\lambda_{t-2} + \cdots + N_{k+1}\lambda_{t-k} + \cdots + \tfrac{1}{2}N_{(t+1)/2}\lambda_{(t+1)/2} \\ + \text{(polynomial in lower orders of } \lambda), \tag{73}$$

where

$$N_{k+1} = \frac{\epsilon_1^k(k+1) - (t-k)}{\epsilon_1^{k+1} - \epsilon_1}\lambda_{k+1} + \text{(polynomial in lower orders of } \lambda). \tag{74}$$

If $k \equiv 0 \pmod{m}$, then the term of order $k+1$ in (74) is missing. If $t = m+1$, every term is present. The relations (72) and (73) are simply more explicit forms of the necessary conditions (32). In similar fashion, we may now write out the corresponding expressions in the μ's, thus obtaining the necessary relations among the coefficients of f and g in (43).

15. We shall now write (53) in a form corresponding to (72) and (73), and then apply once more the method of Sections 12 and 13. Referring to the identity (47) we may write, if t is *even*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,k+1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,t/2}\lambda_{(t/2)+1} + Q_{t,\,t/2}\mu_{(t/2)+1} \\ & + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2)+1), \end{aligned} \tag{75}$$

and if t is *odd*,

$$\begin{aligned} K_t = {} & \epsilon_2\lambda_t + \epsilon_1^t\mu_t \\ & + P_{t,\,2}\lambda_{t-1} + Q_{t,\,2}\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,k+1}\lambda_{t-k} + Q_{t,\,k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + P_{t,\,(t-1)/2}\lambda_{(t+3)/2} + Q_{t,\,(t-1)/2}\mu_{(t+3)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} \\ & + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2), \end{aligned} \tag{76}$$

where

$$P_{t,\,k+1} = (k+1)\epsilon_1^k\mu_{k+1} + \text{(lower orders of } \lambda \text{ and } \mu) \tag{77}$$

and

$$Q_{t,\,k+1} = (t-k)\epsilon_1^{t-k-1}\lambda_{k+1} + \text{(lower orders of } \lambda). \tag{78}$$

All λ's and μ's in (75) and (76) are independent except those of order $pm+1$ or $sn+1$. We now examine the cases determined by (56), (60),

(61), and (62). In that event λ_t and μ_t in (75) and (76) must be replaced by the equivalent expressions of the form (72) or (73). If $t = n + 1$, then all the remaining λ's and μ's in (75) or (76) are independent. The coefficient of λ_{k+1} in (78) becomes $(t - k)\epsilon_1^{-k}$, or $(t - k)\epsilon_2^k$.

Substituting for λ_t and μ_t in (75) their equivalent expressions, we have for t even

$$\begin{aligned} K_t = {} & E_2\lambda_{t-1} + F_2\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + E_{t/2}\lambda_{(t/2)+1} + F_{t/2}\mu_{(t/2)+1} \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t/2) + 1), \end{aligned} \tag{79}$$

and from (76) for t odd

$$\begin{aligned} K_t = {} & E_2\lambda_{t-1} + F_2\mu_{t-1} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + E_{k+1}\lambda_{t-k} + F_{k+1}\mu_{t-k} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + E_{(t-1)/2}\lambda_{(t+3)/2} + F_{(t-1)/2}\mu_{(t+3)/2} \\ & + \left(\frac{\epsilon_2}{2} N_{(t+1)/2}\lambda_{(t+1)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} + \frac{\epsilon_1}{2} T_{(t+1)/2}\mu_{(t+1)/2}\right) \\ & + (\text{polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2), \end{aligned} \tag{80}$$

where

$$E_{k+1} = P_{t,\,k+1} + \epsilon_2 N_{k+1}; \qquad F_{k+1} = Q_{t,\,k+1} + \epsilon_1 T_{k+1}, \tag{81}$$

where T is the expression in μ's corresponding to N in (74), and has the form

$$T_{k+1} = \frac{\epsilon_2^k(k+1) - (t-k)}{\epsilon_2^{k+1} - \epsilon_2}\,\mu_{k+1} + (\text{polynomial in lower orders of } \mu). \tag{82}$$

16. From the form of (75) and (76) we may write

$$\begin{aligned} K_{t-1} = {} & \epsilon_2\lambda_{t-1} + \mu_{t-1} \\ & + P_{t-1,\,2}\lambda_{t-2} + Q_{t-1,\,2}\mu_{t-2} \\ & + P_{t-1,\,3}\lambda_{t-3} + Q_{t-1,\,3}\mu_{t-3} \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot, \end{aligned} \tag{83}$$

and similar expressions for the lower orders of K.

Comparing (83) with (79) or (80), we observe that K_t may be taken independent of the lower orders of K unless

$$\begin{vmatrix} E_2 & F_2 \\ \epsilon_2 & 1 \end{vmatrix} = 0 \tag{84}$$

identically. As already seen, this reduces to (57), involving (60), (61) and (62). Assuming (84) satisfied, and eliminating λ_{t-1} and μ_{t-1} from the expressions for K_t and K_{t-1}, the eliminant and the expression for K_{t-2} can always be satisfied by a proper choice of the λ's and μ's involved unless λ_{t-2} and μ_{t-2} can be eliminated simultaneously. This obviously requires that the coefficients of λ_{t-1}, μ_{t-1}, λ_{t-2}, and μ_{t-2} in the expressions for K_t, K_{t-1} and K_{t-2} shall be linearly dependent. The condition for this is that all 3-rowed determinants of the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} \end{matrix} \right\| \tag{85}$$

vanish.

Assuming (84) satisfied, the second column of (85) may be replaced by 0's, leaving but one determinant for consideration. Since E_3 and F_3 are the only expressions containing λ's and μ's of order 3, we have as a necessary condition, on expanding according to the minors of the elements of the first column and using (62),

$$\epsilon_1^{-1}E_3 - \epsilon_2 F_3 + (\text{polynomial in } \lambda_2) = 0,$$

the λ-polynomial being in fact λ_2^2 multiplied by a constant. Using (74), (82), (77), (78) and (81), this reduces to

$$\frac{t-5}{\epsilon_1^2 - 1}(\epsilon_2\lambda_3 + \epsilon_1^3\mu_3) + (\text{polynomial in } \lambda_2) = 0. \tag{86}$$

Here λ_2 is completely independent and λ_3 and μ_3 restricted only by the relation (50). Assuming for the moment that $t \neq 5$, and using (50), the last equation becomes

$$K_3 + (\text{polynomial in } \lambda_2) = 0. \tag{87}$$

Here K_3 is assigned. Hence λ_2 may be so chosen that (88) is *not* satisfied, unless $K_3 = 0$ and the polynomial in λ_2 in (87) vanishes identically. Both of these conditions are necessary if there is to be a necessary relation among the K's. In Section 13 we found that the polynomial of (87) does vanish identically.

17. Assuming now $K_2 = K_3 = 0$, and following the same reasoning, λ_{t-1}, λ_{t-2}, λ_{t-3} and the corresponding μ's may be chosen so that the conditions for K_t, K_{t-1}, K_{t-2} and K_{t-3} are all satisfied unless the coefficients of the λ's and μ's in these expressions are linearly dependent. As before, the condition for this is that the matrix

$$\left\| \begin{matrix} E_2 & F_2 & E_3 & F_3 & E_4 & F_4 \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & P_{t-1,\,3} & Q_{t-1,\,3} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & P_{t-2,\,2} & Q_{t-2,\,2} \\ 0 & 0 & 0 & 0 & \epsilon_2 & \epsilon_1^{-2} \end{matrix} \right\| \tag{88}$$

be of rank < 4. Since in (84) and (85) the last columns are linearly expressible in terms of the previous columns, we may replace columns 2 and 4 in (88) by 0's, leaving but one 4-rowed determinant for consideration. In (88), E_4 and F_4 are the only elements involving λ's and μ's of order 4. Expanding in terms of the determinants of the last two columns, and replacing μ_2 and μ_3 by their equivalent expressions in λ by putting $K_2 = K_3 = 0$ in (49) and (50),

$$\epsilon_2^2 \begin{vmatrix} E_4 & F_4 \\ \epsilon_2 & \epsilon_1^{-2} \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0.$$

This reduces to

$$\frac{t-7}{\epsilon_1^3 - 1} (\epsilon_2 \lambda_4 + \epsilon_1^4 \mu_4) + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0. \tag{89}$$

If $t \neq 7$, this becomes, on using (51),

$$K_4 + (\text{polynomial in } \lambda\text{'s of order } < 4) = 0. \tag{90}$$

As before, the λ's can be chosen so that (90) is not satisfied, unless $K_4 = 0$ and the λ-polynomial vanishes identically. If $K_4 = 0$, (51) gives μ_4 also in terms of λ's.

Suppose now that the condition that K_t depend for its value on K's of lower order requires

$$K_2 = K_3 = K_4 = \cdots = K_k = 0. \tag{91}$$

These determine $\mu_2, \mu_3, \cdots, \mu_k$ as polynomials in λ's of corresponding and lower orders. Continuing as before, we find as another necessary condition that the matrix

$$\begin{Vmatrix} E_2 & F_2 & E_3 & F_3 & \cdots & E_{k+1} & F_{k+1} \\ \epsilon_2 & 1 & P_{t-1,\,2} & Q_{t-1,\,2} & \cdots & P_{t-1,\,k} & Q_{t-1,\,k} \\ 0 & 0 & \epsilon_2 & \epsilon_1^{-1} & \cdots & P_{t-2,\,k-1} & Q_{t-2,\,k-1} \\ 0 & 0 & 0 & 0 & \cdots & P_{t-3,\,k-2} & Q_{t-3,\,k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \epsilon_2 & \epsilon_1^{1-k} \end{Vmatrix} \tag{92}$$

be of rank $< k + 1$. As before, every second column except the last may be replaced by 0's, leaving but a single $(k + 1)$-rowed determinant for consideration. Expanding according to the elements of the last two columns we obtain

$$\epsilon_2^{k-1} \begin{vmatrix} E_{k+1} & F_{k+1} \\ \epsilon_2 & \epsilon_1 \end{vmatrix} + (\text{polynomial in } \lambda\text{'s of order } < k+1) = 0,$$

which reduces to

$$\frac{t-2k-1}{\epsilon_1^k - 1} (\epsilon_2 \lambda_{k+1} + \epsilon_1^{k+1} \mu_{k+1}) + (\text{polynomial in } \lambda\text{'s of order } < k+1) = 0,$$

or

$$\frac{t-2k-1}{\epsilon_1^k - 1} K_{k+1} + (\text{polynomial in } \lambda\text{'s of order} < k+1) = 0. \tag{93}$$

18. We shall now first restrict ourselves to the case $t = n + 1$. Hence all the terms indicated in (74) and (82) are present, and all λ's and μ's in (75) or (76), except λ_t and μ_t, are independent. We shall find it convenient to consider two cases, according as t is *even* or *odd*.

We shall first consider t *even*. Then $t \neq 2k + 1$ in any case. From the form of (79), we may continue the method of the previous sections up to the case $k = t/2$. Hence, if the condition $K_2 = K_3 = \cdots = K_k = 0$ has been shown necessary, we obtain from (93) the condition $K_{k+1} = 0$ and the further condition that the polynomial in λ vanish identically. Hence we have as a necessary condition that K_t be conditioned in terms of lower orders of K

$$K_2 = K_3 = \cdots = K_{t/2} = 0. \tag{94}$$

If the polynomials indicated in (87) and (90) and that resulting from (93) do not vanish, then of course all K's are independent. However, we shall presently show by an indirect method that they all vanish identically. From (94) and the relations of Section 12, we have all μ's up to order $t/2$ determined as polynomials in λ's of corresponding and lower orders.

Conditions (94) and the vanishing of the polynomials considered above suffice for the elimination of the λ's and μ's of orders $(t/2) + 1$ to $t - 1$ from the expressions for K_t, K_{t-1}, $\cdots$, $K_{(t/2)+1}$. From the method of procedure in Section 13 we observe that, if we eliminate λ_{t-1} and μ_{t-1} from between K_t and K_{t-1} and then λ_{t-2} and μ_{t-2} from between the result and K_{t-2}, etc., the result of the entire elimination of λ's and μ's of orders $t - 1$ down to order $(t/2) + 1$ takes the form

$$K_t + H_2 K_{t-1} + H_3 K_{t-2} + \cdots + H_{t/2} K_{(t/2)+1} = P(\lambda), \tag{95}$$

where $P(\lambda)$ is a polynomial in λ's of order $< (t/2) + 1$. It is also clear that no λ of order $> t/2$ appears in the left member of (95). From the form of (79), (83), etc., we observe that the highest orders of λ and μ in the coefficients E and F following λ_{t-1} and μ_{t-1} are not affected by the elimination of these variables; that those following λ_{t-2} and μ_{t-2} are not affected by the next elimination, etc. In other words, the term of highest order in H_{k+1} in (95) is ϵ_1 times the corresponding term in the coefficient of λ_{t-k} in (79). In other words,

$$H_{k+1} = \frac{t-2k-1}{\epsilon_1^{k+1} - \epsilon_1} \lambda_{k+1} + (\text{polynomial in } \lambda\text{'s of lower order}). \tag{96}$$

The coefficient of λ_{k+1} in (96) is, furthermore, not zero. In the left member of (95), $\lambda_{t/2}$ occurs only in $H_{t/2}$. If it occurs in $P(\lambda)$, it is multiplied by some variable λ. Hence if $K_{(t/2)+1} \neq 0$, the λ's of order $< t/2$ may be assigned arbitrarily and $\lambda_{t/2}$ assigned a value to satisfy (95) for *all* values of the K's. Hence for a *necessary* relation to obtain among the K's, we must have $K_{(t/2)+1} = 0$. Similarly, we must have $K_{(t/2)+2} = 0$, etc. Hence the only way in which a necessary condition must subsist among the K's is to have

$$K_{(t/2)+1} = K_{(t/2)+2} = \cdots = K_{t-1} = 0.$$

If, now, $P(\lambda)$ in (95) vanish identically, the necessary resulting condition would be

$$K_t = 0. \tag{97}$$

Hence, if t is even and $= n + 1$, the only condition that may be required among the K's is to have $K_t = 0$, and this can result only if

$$K_2 = K_3 = \cdots = K_{t-1} = 0,$$

and $P(\lambda)$ vanish identically.

These are, furthermore, sufficient conditions if the polynomials resulting from (93) also vanish identically. If the procedure be followed out in detail for $t = 4$, we obtain (95) in the form

$$K_4 + \frac{\epsilon_1^2\lambda_2}{\epsilon_1 - 1} K_3 = 0. \tag{98}$$

For the case $t = 6$ we obtain, without any essential difficulty,

$$K_6 + \frac{3\lambda_2}{\epsilon_1^2 - \epsilon_1} K_5 + \left[\frac{\epsilon_1^4}{\epsilon_1^2 - 1}\lambda_3 + \frac{2\epsilon_1^3(\epsilon_1 + 2)}{(\epsilon_1 - 1)(\epsilon_1^2 - 1)}\lambda_2^2\right] K_4 = 0. \tag{99}$$

19. Suppose now that t is *odd*. The method of Section 17 can be applied so long as t remains $> 2k + 1$. Hence we obtain as necessary conditions for dependence among the K's

$$K_2 = K_3 = \cdots = K_{(t-1)/2} = 0.$$

Also, from the preceding discussion, the elimination of K's of all orders down to $K_{(t+3)/2}$ does not affect the leading terms of order $(t + 1)/2$ in (80). Evaluating the corresponding expressions in (80),

$$\begin{aligned} \frac{\epsilon_2}{2} N_{(t+1)/2}\lambda_{(t+1)/2} + Q_{t,\,(t+1)/2}\mu_{(t+1)/2} + \frac{\epsilon_1}{2} T_{(t+1)/2}\mu_{(t+1)/2} \\ = \frac{t+1}{4}(\epsilon_2\lambda_{(t+1)/2} - \epsilon_1\mu_{(t+1)/2})^2 + \lambda_{(t+1)/2}\text{ (lower orders)} \\ + \mu_{(t+1)/2}\text{ (lower orders)}. \end{aligned} \tag{100}$$

From the form of (75) and (76),

$$K_{(t+1)/2} = \epsilon_2\lambda_{(t+1)/2} - \epsilon_1\mu_{(t+1)/2} \\ + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \tag{101}$$

Hence the result of the elimination of K's down to order $(t+3)/2$ takes the form

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} \\ = \lambda_{(t+1)/2}\text{ (lower orders)} + \mu_{(t+1)/2}\text{ (lower orders)} \\ + \text{(polynomial in } \lambda\text{'s and } \mu\text{'s of orders} < (t+1)/2). \tag{102}$$

Here $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ are not independent, being connected by (101). The question now arises, can these variables be eliminated from between (102) and (101)? For this purpose we have to evaluate the coefficients of $\lambda_{(t+1)/2}$ and $\mu_{(t+1)/2}$ in (102).

20. We shall, for convenience, introduce

$$e = \frac{t+1}{2}, \tag{103}$$

whence

$$t = 2e - 1; \qquad \epsilon_1^{e-1} = \epsilon_2^{e-1} = -1. \tag{104}$$

We shall now trace back all products of the form $\lambda_e\lambda_{e-1}$, $\lambda_e\mu_{e-1}$, $\mu_e\lambda_{e-1}$, $\mu_e\mu_{e-1}$, occurring in the process of obtaining (102). In the following, all expressions involving such products will be included.

From (44), we have the general form

$$\begin{aligned} \epsilon^t a_t = {} & \epsilon a_t + \lambda_t \\ & + \lambda_2[2a_{t-1} + \cdots + 2a_{e-1}a_{e+1}] \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{e-1}[(e-1)a_{e+1} + (e-1)(e-2)a_2a_e + \text{(lower orders)}] \\ & + \lambda_e[ea_e + e(e-1)a_2a_{e-1} + \text{(lower orders)}] \\ & + \lambda_{e+1}[(e+1)a_{e-1} + \text{(lower orders)}] \\ & + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \lambda_{t-1}[(t-1)a_2]. \end{aligned} \tag{105}$$

From the general form (105) we also have

$$(\epsilon^{t-1} - \epsilon)a_{t-1} = \lambda_{t-1} + \cdots + \lambda_{e-1}[(e-1)a_e + \text{(lower orders)}] \\ + \lambda_e[ea_{e-1} + \text{(lower orders)}] + \cdots; \tag{106}$$

$$(\epsilon^{e+1} - \epsilon)a_{e+1} = \lambda_{e+1} + \lambda_2[2a_e + \text{(lower orders)}] + \cdots + \lambda_e ea_2; \tag{107}$$

$$(\epsilon^e - \epsilon)a_e = \lambda_e + \lambda_2[2a_{e-1} + \text{(lower orders)}] + \cdots \\ + \lambda_{e-1}(e-1)a_2; \tag{108}$$

$$(\epsilon^{e-1} - \epsilon)a_{e-1} = \lambda_{e-1} + \text{(lower orders)}. \tag{109}$$

Formulas (105) and (107) presuppose $t > 5$. The case $t = 5$ will be considered specially.

Using (106), (107), (108), (109) and (45), we have from (105)

$$\begin{aligned}\lambda_t = \lambda_2\lambda_{t-1}&\left[\frac{2}{\epsilon_1 - 1} - \frac{t-1}{\epsilon_1(\epsilon_1 - 1)}\right]\\ &+ \cdots + \lambda_{e-1}\lambda_{e+1}\left[\frac{e}{\epsilon_1} + \frac{\epsilon_1 - 1}{\epsilon_1(\epsilon_1 + 1)}\right]\\ &+ \frac{e}{2\epsilon_1}\lambda_e^2 + \lambda_2\lambda_e\lambda_{e-1}\left[\frac{2e(e-1)}{\epsilon_1^2(\epsilon_1 - 1)} - \frac{2e(2\epsilon_1 + 1) - 4}{\epsilon_1(\epsilon_1^2 - 1)}\right] + \cdots.\end{aligned} \tag{110}$$

We may now proceed to the corresponding expression for μ_t. Since K_{e-1} is presumed zero,

$$0 = \epsilon_2\lambda_{e-1} - \mu_{e-1} + \text{(lower orders)},$$

whence

$$\mu_{e-1} = \epsilon_2\lambda_{e-1} + \text{(lower orders of } \lambda). \tag{111}$$

We may also write

$$K_{e+1} = [\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}] - \lambda_2[2\epsilon_2^2\lambda_e + e\mu_e] + \text{(lower orders } \lambda), \tag{112}$$

$$K_e \;\; = [\epsilon_2\lambda_e - \epsilon_1\mu_e] - (e+1)\epsilon_2^2\lambda_2\lambda_{e-1} + \text{(lower orders } \lambda), \tag{113}$$

$$\begin{aligned}K_e^2 \;\; = [\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 - 2\epsilon_2^2(e+1)\lambda_2\lambda_{e-1}&[\epsilon_2\lambda_e - \epsilon_1\mu_e]\\ &+ \text{(lower orders } \lambda).\end{aligned} \tag{114}$$

The expression for μ_t becomes, since $\epsilon_1\epsilon_2 = 1$,

$$\begin{aligned}\mu_t = \lambda_2\mu_{t-1}&\left[\frac{2\epsilon_2^2}{\epsilon_1 - 1} - \frac{\epsilon_2(t-1)}{\epsilon_1 - 1}\right] + \cdots + \lambda_{e-1}\lambda_{e+1}\left[e - \frac{\epsilon_1 - 1}{\epsilon_1 + 1}\right]\\ &+ \frac{\epsilon_1 e}{2}\mu_e^2 + \epsilon_2\lambda_2\lambda_{e-1}\mu_e\left[\frac{2e(e-1)}{\epsilon_1 - 1} - \frac{2e(2\epsilon_2 + 1) - 4}{\epsilon_1^2 - 1}\right] + \cdots.\end{aligned} \tag{115}$$

21. Referring now to (47), we may write

$$\begin{aligned}K_t = [\epsilon_2\lambda_t + \epsilon_1\mu_t] &+ [-2\epsilon_2^2\lambda_2\lambda_{t-1} + \epsilon_2(t-1)\lambda_2\mu_{t-1}] + \cdots\\ &- \epsilon_2^3\lambda_2[2\lambda_{e-1}\lambda_e + \cdots] + \cdots\\ &+ \mu_{e+1}[(e+1)\epsilon_1^e\lambda_{e-1} + \cdots]\\ &+ \mu_e[e\epsilon_1^{e-1}\lambda_e + e(e-1)\epsilon_1^{e-2}\lambda_2\lambda_{e-1} + \cdots]\\ &+ \mu_{e-1}[(e-1)\epsilon_1^{e-2}\lambda_{e+1} + (e-1)(e-2)\epsilon_1^{e-3}\lambda_2\lambda_e + \cdots] + \cdots,\end{aligned}$$

which may be written

$$\begin{aligned}K_t = [\epsilon_2\lambda_t + \epsilon_1\mu_t] &+ \lambda_2[-2\epsilon_2^2\lambda_{t-1} + \epsilon_2(t-1)\mu_{t-1}] + \cdots\\ &+ \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_{e+1} - \epsilon_1(e+1)\mu_{e+1}] - e\lambda_e\mu_e\\ &+ \lambda_2\lambda_{e-1}\{\lambda_e[-2\epsilon_2^3 - \epsilon_2^3(e-1)(e-2)]\\ &\qquad + \mu_e[-\epsilon_2 e(e-1)]\} + \cdots.\end{aligned} \tag{116}$$

The only other K involving like products is

$$K_{t-1} = [\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots + \lambda_{e-1}[-\epsilon_2^2(e-1)\lambda_e - e\lambda_e] + \cdots. \tag{117}$$

Substituting now from (110) and (115) in (116),

$$K_t = -\epsilon_2 \frac{t-3}{\epsilon_1 - 1}\lambda_2[\epsilon_2\lambda_{t-1} + \mu_{t-1}] + \cdots$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1+1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-\frac{\epsilon_1-3}{\epsilon_1-1}e^2 - \frac{\epsilon_1^2+4\epsilon_1+5}{\epsilon_1^2-1}e \right. \\ \qquad \left. + \frac{4}{\epsilon_1^2-1}(-\epsilon_1^2+\epsilon_1+1)\right] \\ + \epsilon_1\mu_e\left[\frac{\epsilon_1+1}{\epsilon_1-1}e^2 - \frac{\epsilon_1^2+4\epsilon_1+5}{\epsilon_1^2-1}e \right. \\ \qquad \left. + \frac{4\epsilon_1}{\epsilon_1^2-1}\right] \end{array}\right\} + \cdots. \tag{118}$$

From (118) and (117), we now have

$$K_t + \frac{\epsilon_2(t-3)}{\epsilon_1-1}\lambda_2 K_{t-1} = [\text{orders} < t-1 \text{ and } > e+1]$$
$$+ \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + \frac{2\lambda_{e-1}}{\epsilon_1+1}[\epsilon_2\lambda_{e+1} - \epsilon_1^2\mu_{e+1}]$$
$$+ \epsilon_2^2\lambda_2\lambda_{e-1}\left\{\begin{array}{l} \epsilon_2\lambda_e\left[-e^2 - \frac{\epsilon_1^2-2\epsilon_1-1}{\epsilon_1^2-1}e \right. \\ \qquad \left. - \frac{4\epsilon_1^2}{\epsilon_1^2-1}\right] \\ + \epsilon_1\mu_e\left[e^2 - \frac{\epsilon_1^2+1}{\epsilon_1^2-1}e + \frac{4\epsilon_1}{\epsilon_1^2-1}\right] \end{array}\right\} + \cdots. \tag{119}$$

We may observe that the right member of (119) is not affected by the elimination of any more λ's and μ's down to those of order $e+1$. From (96),

$$H_{(t-1)/2} = -\frac{2\lambda_{e-1}}{\epsilon_1+1} + (\text{lower orders of } \lambda). \tag{120}$$

In the process of obtaining (102) we have then, from the form (119) and (112),

$$K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2}$$
$$= \frac{e}{2}[\epsilon_2\lambda_e - \epsilon_1\mu_e]^2 + [\text{polynomial in } \lambda\text{'s of order} < e] \tag{121}$$
$$- \frac{\epsilon_2^2\lambda_2\lambda_{e-1}}{\epsilon_1^2-1}[e^2(\epsilon_1^2-1) + (\epsilon_1^2-2\epsilon_1-1)e + 4\epsilon_1][\epsilon_2\lambda_e - \epsilon_1\mu_e].$$

From (121) and (114),

$$\begin{aligned} K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} \\ = \epsilon_2\frac{t-3}{\epsilon_1^2-1}\lambda_2\lambda_{e-1}[\epsilon_2\lambda_e - \epsilon_1\mu_e] + (\text{polynomial in } \lambda\text{'s of order} < e). \end{aligned} \tag{122}$$

Using (113), (122) may now be written

$$\begin{aligned} K_t + H_2K_{t-1} + \cdots + H_{(t-1)/2}K_{(t+3)/2} - \frac{t+1}{4}K^2_{(t+1)/2} \\ - \epsilon_2\frac{t-3}{\epsilon_1^2-1}\lambda_2\lambda_{(t-1)/2}K_{(t+1)/2} = P(\lambda), \end{aligned} \tag{123}$$

where $P(\lambda)$ is some polynomial in λ's of order $< (t+1)/2$. Equation (123) corresponds to (102). For the special case $t = 5$, we obtain readily

$$K_5 + (\epsilon_1 - 1)\lambda_2K_4 - \frac{3}{2}K_3^2 - \frac{\epsilon_1}{2}\lambda_2^2K_3 = 0. \tag{124}$$

In (123), $\lambda_{(t-1)/2}$ occurs in the left member only in the coefficients of $K_{(t+3)/2}$ and $K_{(t+1)/2}$. In the latter it occurs in product with the independent λ_2. Hence, reasoning as before, (123) can be made to hold for all K's, unless $K_{(t+1)/2} = 0$ and $K_{(t+3)/2} = 0$. Similarly, we must require

$$K_{(t+5)/2} = \cdots = K_{t-1} = 0.$$

Only in that event, and if $P(\lambda) = 0$ identically, can we obtain the necessary condition $K_t = 0$.

We have seen that if $K_1 \neq 1$ in (43), then F is always factorable as the product of two periodic transformations. If $K_1 = 1$ and F is factorable, the periods of f and g in (43) must be equal, giving $m = n$. We may now state the result

If the transformation

$$F(z) = z + K_2z^2 + K_3z^3 + \cdots$$

is to be factored into two transformations of period n, then the only condition that may be found necessary among the K's is of the form

$$K_{sn+1} = \textit{linear expression in } K\textit{'s of lower order;}$$

if s is to $= 1$, *the only way in which the condition can arise is to have*

$$K_2 = K_3 = \cdots = K_n = 0,$$

in which event the condition is

$$K_{n+1} = 0.$$

In other words, if $F(z)$ takes the form

$$z + K_r z^r + \cdots,$$

then F can always be factored into two transformations of period $> r - 1$.

It should be remembered that we have not as yet shown that if the condition

$$K_2 = K_3 = \cdots = K_n = 0 \tag{125}$$

holds, then $K_{n+1} = 0$ necessarily. In other words, we have to show that $P(\lambda)$ in (95) and (123) and the λ-polynomials resulting from (93) all vanish identically. We have besides to consider the more general case where $t = sn + 1$, $s > 1$.

22. We shall consider the former question first. It should be observed that $P(\lambda)$ in (95) and (123), and the other λ-polynomials we have presumed to vanish are independent of K_{n+1} or any K of higher order. Hence, if it can be shown that the function

$$y = z + z^{n+1}; \qquad n > 2 \tag{126}$$

cannot be factored into two transformations of period n, then it would follow that $P(\lambda) = 0$ identically, and that the other λ-polynomials vanish correspondingly.

23. We have seen, Theorem V, that if in (36)

$$A_1 = 1, \qquad A_{kn+1} = 0; \qquad k = 1, 2, 3, \cdots,$$

then there is one and only one transformation (36) which will put (34) into the form (35). Suppose now that F in (43) takes the form (126) and we require f and g as defined in (43) to satisfy

$$g[f(z)] = z + z^{n+1}. \tag{127}$$

As we have seen, we must have $\epsilon_1\epsilon_2 = 1$, and $m = n$. Furthermore, from (43) and (127) we observe, on writing out the detailed conditions on λ's and μ's, that each coefficient μ is determined uniquely as a polynomial in λ's of corresponding and lower orders. Hence, *for all orders* $< n + 1$, the coefficients μ have the same relations to the coefficients λ as would be obtained from (43) and the condition

$$g[f(z)] = z. \tag{128}$$

This condition would require $g = f^{-1}$. Furthermore, there exists a unique function

$$h(z) = z + \sum_{s=0}^{\infty} \sum_{r=1}^{n-1} A_{sn+r+1} z^{sn+r+1} \tag{129}$$

such that, symbolically,

$$f = h^{-1}\epsilon_1 h. \tag{130}$$

From (130), $f^{-1} = h^{-1}\epsilon_1^{-1}h$. Hence $g(z)$ as determined by (128) may be written

$$g = h^{-1}\epsilon_1^{-1}h. \tag{131}$$

The first n coefficients of g as determined by (131) are identical with the corresponding coefficients as determined by (127). Furthermore, since f and g in (127) are periodic functions of order n, the coefficients λ_{n+1} and μ_{n+1} are determined necessarily as polynomials of the coefficients of lower order. Furthermore, the conditions are identical with those arising from (130) and (131) for λ_{n+1} and μ_{n+1} respectively. Hence the forms (130) and (131) for f and g hold for coefficients λ and μ of all orders *up to and inclusive of* $n + 1$. Hence the product is valid for powers of the variable including the $(n + 1)$th. But

$$h^{-1}\epsilon_1 h \cdot h^{-1}\epsilon_1^{-1}h = 1.$$

Hence, necessarily,

$$g[f(z)] = z + (\text{powers of } z > n + 1). \tag{132}$$

Obviously (132) is inconsistent with (127), but results necessarily from the assumption that K_1 in (43) $= 1$, that f and g are of period n and that conditions (125) hold.

Hence $z + z^{n+1}$ cannot be factored into transformations of period n. Hence $P(\lambda)$ in (95) and (123) must $= 0$ identically, and all the other conditions for a necessary relation among the K's must be satisfied identically. Hence

If $z + K_{n+1}z^{n+1} + \cdots$ *is to be factored into two transformations of period* n, *we have as a necessary condition*

$$K_{n+1} = 0.$$

24. Suppose now in the discussion preceding Section 18, we choose $t = 2n + 1$. Then t is *odd*. Furthermore, only the coefficients of λ's and μ's of order $n + 1$ in (74), (82) and (76) will be affected. All coefficients λ and μ between the orders $n + 1$ and $2n + 1$ remain independent. The method of Sections 17 and 19 then holds valid so long as

$$k < \frac{t-1}{2}, \qquad \text{or} \qquad k < n,$$

yielding the necessary conditions

$$K_2 = K_3 = \cdots = K_n = 0.$$

From the previous section this requires in any case

$$K_{n+1} = 0,$$

determining μ_{n+1} in terms of λ's of corresponding and lower orders, hence in terms of λ's of order $< n + 1$. The resulting relation among the K's becomes

$$K_{2n+1} + H_2K_{2n} + \cdots + H_nK_{n+2} = P(\lambda), \tag{133}$$

where $P(\lambda)$ is a polynomial in λ's of order $< n + 1$, replacing λ_{n+1} by its equivalent in terms of λ's of lower order. Hence if

$$K_{n+2} = K_{n+3} = \cdots = K_{2n} = 0,$$

and $P(\lambda)$ vanishes identically, we obtain the necessary condition

$$K_{2n+1} = 0. \tag{134}$$

This presupposes, too, that the λ-polynomials occurring in the process of obtaining (133) all vanish identically. Hence if

$$K_2 = K_3 = \cdots = K_{2n} = 0, \tag{135}$$

then $K_{2n+1} = 0$.

Following identically the same reasoning as above we have for $t = 3n + 1$, if (135) hold true,

$$K_t + H_2K_{t-1} + \cdots + H_nK_{t-n+1} = P(\lambda), \tag{136}$$

where $P(\lambda)$ is some polynomial in λ's of order $< t - n$. Hence again, if $P(\lambda) = 0$ and $K_{t-n+1} = \cdots = K_{t-1} = 0$, we must have $K_t = 0$.

The reasoning is clearly general, and we may put $t = sn + 1$ in (136). Hence if

$$K_2 = K_3 = \cdots = K_{sn} = 0, \tag{137}$$

we must have as a necessary condition

$$K_{sn+1} = 0. \tag{138}$$

Furthermore, we observe that the reasoning of Section 23 is general, and that if (137) hold true, then (138) must be satisfied. This is, furthermore, the only way in which a necessary condition may be required among the coefficients K.

We may now state the complete

THEOREM VI. *If the transformation*

$$F(z) = K_1z + K_2z^2 + \cdots$$

is to be factorable into two periodic transformations, we must have

$$|K_1| = 1,$$

with a commensurable argument; if $K_1 \neq 1$, the factorization is always possible, the period being so taken that K_1 = the product of the leading coeffi-

cients of the factor transformations; if $K_1 = 1$, *the periods of the factor transformations, if they exist, must be equal; if* $K_1 = 1$, *and* $K_2 \neq 0$, *then* $F(z)$ *is always factorable into two transformations of arbitrary period* > 2; *if* $K_1 = 1$ *and*

$$K_2 = K_3 = \cdots = K_r = 0; \qquad K_{r+1} \neq 0,$$

then $F(z)$ *can not be factored into transformations of order* r *or any factor of* r, *but can always be factored into transformations of any other order* > 2.

25. For example, if

$$F(z) = z + z^{13},$$

$F(z)$ cannot be factored into transformation of period 2,* 3, 4, 6 or 12. It can, however, be factored into transformations of period 5, 7, 8, 9, 10, 11, or any period > 12. On the other hand,

$$F(z) = -z + z^{13}$$

can be factored as the product of transformations of any even period > 2.

It may be observed, also, that in any case where factorization is possible, transformations with equal irreducible periods may be chosen. This is evident from the single restriction (48).

All transformations

$$F(z) = \epsilon z + K_2 z^2 + K_3 z^3 + \cdots; \qquad \epsilon^n = 1$$

are factorable into periodic transformations and constitute a group, the group generated by all periodic transformations. The class defined by $\epsilon = 1$ constitutes a subgroup. The class defined by $\epsilon = 1$, $K_2 = 0$ is a subgroup of the last. The class defined by $\epsilon = 1$, $K_2 = K_3 = 0$ is a subgroup of the previous, and so on. In any of the previous cases, the class defined by $\epsilon = 1$,

$$K_2 = K_3 = \cdots = K_r = 0 \qquad \text{and} \qquad K_{r+1} \neq 0$$

constitutes a subgroup. The latter may be characterized by the index of the highest period of transformations (r) into which $F(z)$ cannot be factored.

It should be observed further that, though

$$F(z) = z + K_{r+1} z^{r+1} + \cdots; \qquad r > 2 \tag{139}$$

cannot be factored into *two* transformations of period r, it can always be factored into *three* transformations of period r. Further, (139) can always be factored into a rotation through a rational angle and two periodic transformations. The latter, too, can be taken of period r, by choosing the corresponding angle.

* See Kasner, loc. cit.

REVERSE CONFORMAL TRANSFORMATIONS (CONFORMAL SYMMETRIES).

26. Kasner also considers another type of conformal transformation. This he calls the *reverse* or *improper* type of transformation, which he defines by

$$y = f(z_0), \tag{140}$$

where f is non-singular at the origin and z_0 is the conjugate of the variable z. He defines reverse transformations of period 2 as *conformal symmetries* since they are conformally reducible to the form $y = z_0$ (conformally equivalent to Schwarzian reflection) and discusses them in parallel with what he terms the *direct conformal transformations.*

We shall now show that no other types of reverse conformal transformations of regular period exist; in other words, every periodic reverse conformal transformation is of irreducible period 2, that is, a conformal symmetry.

27. Denoting the transform of z, $f(z_0)$, by $F(z)$, so that

$$F(z) = f(z_0), \tag{141}$$

we have

$$F_2(z) = f[f(z_0)]_0,$$

where the zero subscript denotes conjugate values.

We now observe that, in general,

$$\begin{gathered}(x + y)_0 = x_0 + y_0, \quad (xy)_0 = x_0y_0, \quad (z^n)_0 = (z_0)^n, \quad (az^n)_0 = a_0z_0^n, \\ [f(z)]_0 = f_0(z_0), \quad (z_0)_0 = z, \quad [f(z_0)]_0 = f_0(z),\end{gathered} \tag{142}$$

where the coefficients of f_0 are the conjugates of those of f.

Hence

$$F_2(z) = f[f_0(z)]. \tag{143}$$

Similarly,

$$\begin{aligned}F_3(z) &= f\{f[f_0(z)]\}_0 \\ &= f\{f_0[f_0(z)]_0\} \\ &= f\{f_0[f(z_0)]\}.\end{aligned}$$

Hence we may write

$$F_{2k-1}(z) = (ff_0)^{k-1}f(z_0), \tag{144}$$

$$F_{2k}(z) = (ff_0)^k(z). \tag{145}$$

28. We shall consider the case $F_{2k-1}(z) = z$ first. If f and f_0 be defined by

$$\begin{aligned}f(z) &= a_1z + a_2z^2 + \cdots, \\ f_0(z) &= b_1z + b_2z^2 + \cdots,\end{aligned} \tag{146}$$

we must have

$$b_r = \frac{|a_r|^2}{a_r}, \qquad |a_1| = 1. \tag{147}$$

Hence $a_1 b_1 = 1$, and

$$f[f_0(z)] = z + \text{(higher powers)}. \tag{148}$$

Hence from (144) we would have

$$a_1 z_0 + \text{(higher powers of } z_0) = z, \tag{149}$$

for all values of z. On comparing coefficients of real and imaginary components of the variable z, the relation (149) is manifestly impossible. Hence, if

$$F_n(z) = z \tag{150}$$

identically, n must be *even*.

29. Suppose now $F_{2k}(z)$ is identically z in (145). Then

$$(ff_0)^k(z) = z. \tag{151}$$

Here ff_0 is a definite transformation. Putting $g(z) = ff_0(z)$, we have then g a *direct* conformal transformation of period k, and we may write

$$g(z) = ff_0(z) = \epsilon z + \lambda_2 z^2 + \lambda_3 z^3 + \cdots; \qquad \epsilon^k = 1. \tag{152}$$

But by (148) the leading coefficient of $ff_0(z)$ is 1. Hence in (152) we must have $\epsilon = 1$. Hence, by Theorem I,

$$\lambda_2 = \lambda_3 = \cdots = 0,$$

and we have

$$F_2(z) = f[f_0(z)] = z \tag{153}$$

necessarily. Hence in any case F is of period 2. Hence we may state

THEOREM VII. *Every periodic reverse conformal transformation is of period* 2; *in other words, the only kind of periodic reverse conformal transformations are conformal symmetries.*

30. In his discussion of conformal symmetries, Kasner obtains the result that the transformation

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1, \tag{154}$$

is always factorable into two symmetries if the angle of K_1 is *irrational.* From the discussion in this paper, (154) is factorable into two direct periodic conformal transformations if the angle of K_1 is *rational.* Hence we have

THEOREM VIII. *The transformation*

$$K_1 z + K_2 z^2 + K_3 z^3 + \cdots; \qquad |K_1| = 1,$$

is always factorable either into two conformal symmetries or into two direct periodic conformal transformations.

It would seem also that, in general, a direct periodic conformal transformation is not factorable into two symmetries.

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